

ON A SYMMETRIZATION OF DIFFUSION PROCESSES

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ABSTRACT. The latter author, together with collaborators, proposed a numerical scheme to calculate the price of barrier options in [3, 4, 5]. The scheme is based on a symmetrization of diffusion process. The present paper aims to give a mathematical credit to the use of the numerical scheme for Heston or SABR type stochastic volatility models. This will be done by showing a fairly general result on the symmetrization (in multi-dimension/multi-reflections). Further applications (to time-inhomogeneous diffusions/ to time dependent boundaries/to curved boundaries) are also discussed.

1. INTRODUCTION

Recently, the latter author, together with Yuta Ishigaki, Takuya Kawagoe and Toshiaki Okumura, introduced a new numerical scheme for calculating the price of barrier options in a series of papers [3, 4, 5]. The scheme is based on what they call “put-call symmetry”, a notion introduced by Peter Carr and Roger Lee [1] in relation to (a generalization of) static hedging of barrier options.

The put-call symmetry, PCS for short, of a diffusion process X at a point $K \in \mathbf{R}$, means the equivalence in law between $K - X_t$ (put) and $X_t - K$ (call) for arbitrary $t \geq \tau_K$, where τ_K is the first hitting time at K . This is much weaker than the *reflection principle* which has been widely-recognized as a fundamental requirement for the static hedging of barrier options. While the put-call symmetry is still something one cannot expect for a given diffusion without luck, the latter author and her collaborators have noticed that, for any given diffusion

- one can construct (easily) another diffusion that is identical with the diffusion up to the first hitting time and satisfies the put-call symmetry,
- by which the price of barrier-type options written on the original diffusion is expressed by a combination of the prices of plain options written on the constructed one.
- The fact in turn implies that the “symmetrization” offers a new numerical scheme for calculating the price of barrier options; it

transforms path-dependent expectations to path-independent ones.

The numerical experiments they performed show that the scheme is quite plausible. They also claim in [4, 5] that it can be applicable to stochastic volatility models where the stock and its volatility are described by a two-dimensional diffusion.

In this paper, we give a mathematical background for the scheme by establishing some symmetry results in a more general setting under the action of a reflection group. This in turn leads to further possible applications of the scheme. We will show that it can be used for the cases with time-inhomogeneous diffusions, time-dependent boundaries, as well as curved boundaries.

This paper is organized as follows. We start with a detailed discussion of the put-call symmetry (section 2). After recalling the one-dimensional cases (subsection 2.1), we first introduce a multi-dimensional generalization (subsection 2.2) and then extend it to multi-reflections (subsection 2.3). The key assumptions are found in the statement of Lemma 2.2. Section 3 is devoted to discussions on the ‘symmetrization’. Again starting from a review of one-dimensional cases (subsection 3.1), we give the main result (Theorem 3.4) in full generality in subsection 3.2. Applications of the main theorem are presented in section 4. We first give a credit to the use of the scheme for stochastic volatility models including Heston’s and SABR type (subsection 4.1). The trick for the generic stochastic volatility models is extended to get a generalization of Theorem 3.4 (Theorem 4.3), as a corollary to which we show that the scheme is applicable even to time-inhomogeneous diffusions (Corollary 4.4). This observation further enables the applications to curved boundary cases (subsection 4.3) and time-dependent boundary cases (subsection 4.4).

2. THE PUT-CALL SYMMETRY

2.1. One dimensional case revisited. Let X be a one dimensional diffusion satisfying the put-call symmetry at $K \in \mathbf{R}$;

$$(2.1) \quad X_t 1_{\{\tau_K \leq t\}} \stackrel{d}{=} (2K - X_t) 1_{\{\tau_K \leq t\}}$$

for any $t > 0$, where $\tau_K = \inf\{t > 0 : X_t = K\}$. The PCS (2.1) alone, without the reflection principle, suffices to have a static hedging formula for barrier options. For completeness, we give a brief proof on it. Put

$$\tilde{X}_t = X_t 1_{\{\tau_K > t\}} + (2K - X_t) 1_{\{\tau_K \leq t\}}.$$

Then by the PCS (2.1), we have

$$(2.2) \quad X_t \stackrel{d}{=} \tilde{X}_t,$$

for arbitrary $t > 0$. The equivalence in law (2.2) in turn implies

$$(2.3) \quad P(X_t \in A : \tau_K \leq t) = P(\tilde{X}_t \in A : \tau_K \leq t)$$

for any Borel set A . Suppose that $X_0 > K$ and $A \subset \{x > K\}$. Then the right-hand-side is equal to $P(2K - X_t \in A)$ since $\{\tau_K \leq t\}$ is included in $\{2K - X_t \in A\}$. Thus it holds that

$$(2.4) \quad P(X_t \in A : \tau_K > t) = P(X_t \in A) - P(X_t \in 2K - A)$$

for any Borel set A with $A \subset \{x > K\}$ (see [1], [3]). In other words,

$$(2.5) \quad \begin{aligned} & E[f(X_t - K)1_{\{X_t > K\}}1_{\{\tau_K > t\}}] \\ &= E[f(X_t - K)1_{\{X_t > K\}}] - E[f(K - X_t)1_{\{X_t < K\}}] \end{aligned}$$

for any bounded Borel function f and $t > 0$, which can be understood as a static hedging formula.

2.2. A multi-dimensional generalization of PCS. To generalize the argument in the previous subsection, we understand $x \mapsto 2k - x$ as a reflection. In \mathbf{R}^d , a reflection is associated with a hyperplane. A hyperplane is given by $H_{\alpha,k} := \{x \in \mathbf{R}^d : \langle \alpha, x \rangle = k\}$, where $k \in \mathbf{R}$ and $\alpha (\neq 0) \in \mathbf{R}^d$. The reflection $s_{\alpha,k} : \mathbf{R}^d \rightarrow \mathbf{R}^d$ with respect to $H_{\alpha,k}$ is given by

$$s_{\alpha,k}(x) = x - (\langle x, \alpha \rangle - k) \frac{2\alpha}{|\alpha|^2}.$$

Notice that $s_{\alpha,k}^2 = \text{Id}_{\mathbf{R}^d}$. A natural extension of the previous PCS could be as follows.

Definition 2.1. Let X be a diffusion process in \mathbf{R}^d . We say X has the put-call symmetry with respect to the hyperplane $H_{\alpha,k}$ if

$$(2.6) \quad X_t 1_{\{\tau_{\alpha,k} \leq t\}} \stackrel{d}{=} s_{\alpha,k}(X_t) 1_{\{\tau_{\alpha,k} \leq t\}}$$

for any $t > 0$, where $\tau_{\alpha,k} = \inf\{t > 0 : X_t = H_{\alpha,k}\}$.

In a totally similar way as above, we can obtain a static hedging formula corresponding to (2.5). In fact, we have, for $t > 0$,

$$(2.7) \quad E[f(X_t)1_{\{\tau_{\alpha,k} > t\}}] = E[f(X_t)] - E[f(s_{\alpha,k}(X_t))].$$

for any bounded measurable f whose support is included in a half space $\{x \in \mathbf{R}^d | \pm(\langle \alpha, x \rangle - k) > 0\}$, provided that X has the PCS with respect to a hyperplane $H_{\alpha,k}$.

2.3. The PCS with respect to a reflection group. As the reflection principle is generalized to multiple reflections which form a group (see [6] and references therein), so is the put-call symmetry. In this section, we discuss the generalization in detail.

It may be natural that we consider the exit time out of an intersection of hyperplanes. More precisely, denoting

$$H_{\alpha,k}^+ := \{x \in \mathbf{R}^d | \langle \alpha, x \rangle - k > 0\}$$

and

$$\Sigma_\Phi := \bigcap_{(\alpha,k) \in \Phi} H_{\alpha,k}^+,$$

we set, for a given diffusion X ,

$$\tau_{\Sigma_\Phi} := \inf\{t > 0 : X_t \notin \Sigma_\Phi\} = \min_{(\alpha,k) \in \Phi} \tau_{\alpha,k},$$

and consider the problem of representing the expectation of $f(X_t)1_{\{\tau_{\Sigma_\Phi} > t\}}$ by those of $f(g(X_t))$, where g runs through a set G , which will turn out to be the group generated by the reflections s_λ , $\lambda \in \Phi$.

Looking at the discussion in section 2.1, we notice that the key was the equation (2.3), which is not anymore directly applicable to the multi-reflection case. However, we have the following generalization.

Lemma 2.2. *Let G be the group generated by the reflections $\{s_{\alpha,k} : (\alpha,k) \in \Phi\}$ and X be a diffusion process in \mathbf{R}^d satisfying PCS with respect to $H_{\alpha,k}$ for all $(\alpha,k) \in \Phi$. Assume that (i) $g\Sigma \cap g'\Sigma = \emptyset$ whenever $g \neq g' \in G$, and (ii) there is a group homomorphism $\eta : G \rightarrow \mathbf{C}$ (character) such that $\eta(s_{\alpha,k}) = -1$ for each $(\alpha,k) \in \Phi$. Then, for a Borel subset A of Σ , $x \in \Sigma$, and $t > 0$, we have*

$$(2.8) \quad \sum_{g \in G} \eta(g) P_x(X_t \in gA : \tau_{\Sigma_\Phi} \leq t) = 0.$$

Proof. We first note that the left-hand-side of (2.8) is absolutely convergent since the sets $gA, g \in G$ are disjoint. Therefore we can change the order as

$$(2.9) \quad \begin{aligned} & \sum_{g \in G} \eta(g) P_x(X_t \in gA : \tau_{\Sigma_\Phi} \leq t) \\ &= \sum_{(\alpha,k) \in \Phi} \sum_{g \in G} \eta(g) P_x(X_t \in gA : \tau_{\Sigma_\Phi} = \tau_{\alpha,k} \leq t). \end{aligned}$$

By the assumption on the put-call symmetry, we have

$$(2.10) \quad \begin{aligned} P_x(X_t \in gA : \tau_{\Sigma_\Phi} = \tau_{\alpha,k} \leq t) &= P_x(s_{\alpha,k}(X_t) \in gA : \tau_{\Sigma_\Phi} = \tau_{\alpha,k} \leq t) \\ &= P_x(X_t \in s_{\alpha,k}gA : \tau_{\Sigma_\Phi} = \tau_{\alpha,k} \leq t). \end{aligned}$$

On the other hand, by the assumption on η we have

$$\begin{aligned} & \sum_{g \in G} \eta(g) P_x(X_t \in s_{\alpha,k} g A : \tau_{\Sigma_\Phi} = \tau_{\alpha,k} \leq t) \\ &= - \sum_{g \in G} \eta(s_{\alpha,k} g) P_x(X_t \in s_{\alpha,k} g A : \tau_{\Sigma_\Phi} = \tau_{\alpha,k} \leq t), \end{aligned}$$

which is equal to

$$- \sum_{g \in G} \eta(g) P_x(X_t \in g A : \tau_{\Sigma_\Phi} = \tau_{\alpha,k} \leq t),$$

thanks to the group structure. This observation together with the equation (2.10) shows that

$$\sum_{g \in G} \eta(g) P_x(X_t \in g A : \tau_{\Sigma_\Phi} = \tau_{\alpha,k} \leq t) = 0,$$

which proves the assertion with (2.9). \square

Thanks to the lemma, we have a generalization of the static hedging formula.

Theorem 2.3. *We keep the notations and the assumptions of the Lemma 2.2. For a bounded Borel function f with its support in Σ_Φ , we have*

$$(2.11) \quad E[f(X_t) 1_{\{\tau_{\Sigma_\Phi} > t\}}] = \sum_{g \in G} \eta(g) E[f(g^{-1}(X_t))].$$

Proof. It suffices to show that

$$P_x(X_t \in A : \tau_{\Sigma_\Phi} > t) = \sum_{g \in G} \eta(g) P_x(X_t \in g A),$$

but this is equivalent to (2.8) since

$$P_x(X_t \in g A : \tau_{\Sigma_\Phi} > t) = 0$$

unless $g \neq 1$, by the assumption (ii) in Lemma 2.2. \square

Remark 2.4. The one reflection case in section 2.2 automatically satisfies the assumptions in Lemma 2.2, and apparently (2.11) includes (2.7).

Remark 2.5. The case with $\Phi = \{(\alpha, k), (-\alpha, -k - 1)\}$ corresponds to the double boundary reflections, where, by choosing conventionally $|\alpha| = 1$ and η to be the (mod 2) “length” of the reflections (see e.g. [2]),

$$P_x(X_t \in A; \tau_{\Sigma_\Phi} > t) = \sum_{k \in \mathbf{Z}} \{P_x(X_t \in A + 2k\alpha) - P_x(X_t \in -A + (2k+1)\alpha)\}.$$

Remark 2.6. More generally, in the cases of multi-reflections, there do exist the situations where the assumptions are fulfilled. There are two important classes. One is that of finite reflection groups and the other, of affine reflection groups. The domain Σ is a cone in the former class and a simplex in the latter. In fact, if $\{\alpha\}$ forms a so-called *simple* system (or fundamental system) of a *root* system, and (i) if k is fixed, then the group becomes finite, and (ii) if additionally α is taken from a root system properly and k' is taken to be $-k - 1$, then the group becomes (isomorphic) to the semi-direct product of the finite reflection group and the translation group \mathbf{Z} , which is called an *affine reflection group*. In both cases we can take η to be its determinant when the finite reflection group is embedded into the orthogonal group and therefore into the general linear group. For details, see e.g. [2].

3. THE SYMMETRIZATION

Let X be a solution to

$$(3.1) \quad dX_t = \sigma(X_t) dW_t + \mu(X_t) dt,$$

where $\sigma : \mathbf{R}^d \rightarrow \mathbf{R}^d \times \mathbf{R}^d$ and $\mu : \mathbf{R}^d \rightarrow \mathbf{R}^d$ are piecewise continuous functions with at most linear growth, and W denotes a d -dimensional standard Wiener process. We further impose some ellipticity conditions on $\sigma\sigma^*$ to ensure that a unique (weak) solution to (3.1) exists, and that Euler-Maruyama approximation of the solution of (3.1) works;

3.1. One dimensional case revisited. In the case of $d = 1$ (with $\alpha = 1$), Carr and Lee [1] showed that

$$(3.2) \quad \sigma(x) = \pm\sigma(2k - x), \quad \mu(x) = -\mu(2k - x)$$

is a sufficient condition under which the solution X of (3.1) has the PCS. As is pointed out in Introduction, even though this is much weaker than the reflection principle which can be now rephrased as “PCS at any k ”, yet it is not practical to assume a price process to satisfy PCS.

In [3], however, the following observation is made;

Theorem 3.1 ([3]). *Let X be a solution to (3.1) without (3.2) and \tilde{X} be one to*

$$d\tilde{X}_t = \tilde{\sigma}(\tilde{X}_t) dW_t + \tilde{\mu}(\tilde{X}_t) dt,$$

where

$$\tilde{\sigma}(x) = \sigma(x)1_{\{x > k\}} \pm \sigma(2k - x)1_{\{x \leq k\}},$$

and

$$\tilde{\mu}(x) = \mu(x)1_{\{x > k\}} - \mu(2k - x)1_{\{x \leq k\}}.$$

We assume $X_0 = \tilde{X}_0 > K$. Then we have

$$(3.3) \quad \begin{aligned} & E[f(X_t - K)1_{\{X_t > K\}}1_{\{\tau_K > t\}}] \\ &= E[f(\tilde{X}_t - K)1_{\{\tilde{X}_t > K\}}] - E[f(K - \tilde{X}_t)1_{\{\tilde{X}_t < K\}}] \end{aligned}$$

for any bounded Borel function f and $t > 0$.

They claimed that the formula (3.3) gives a new insight to financial engineering of barrier options; it says that the price of barrier option is expressed by those of plain options. Numerical analysis for the former is difficult, while the latter is much easier. In fact, the numerical experiments they performed (part of which is appeared in ([3])) show that their scheme is quiet effective.

Remark 3.2. The PCS method can be seen as a diffusion equation counterpart (or simply a generalization) of so-called *method of image charges* in electrostatics. The authors thank Prof. Sergey Nadtochiy for suggesting to us this “synchronicity”.

3.2. Symmetrization with respect to a reflection group. They also applied their scheme to stochastic volatility models like Heston or SABR type, where above Theorem 3.1 cannot be applied directly since they are basically two dimensional models. With a view to endowing a certificate, we give a general result on multi-dimensional multi-reflection cases in this section. The certificate will be provided in the next section as a corollary to the result.

We start with a lemma.

Lemma 3.3. *Let A be an affine transformation in \mathbf{R}^d such that $Ax = A_0x + a$ for $x \in \mathbf{R}^d$ with $A_0 \in GL(d, \mathbf{R})$ and $a \in \mathbf{R}^d$. Suppose that*

$$(3.4) \quad \sigma(Ax) = A_0\sigma(x)U_x, \quad \mu(Ax) = A_0\mu(x)$$

for $x \in \mathbf{R}^d$, with some piecewise continuous $x \mapsto U_x \in O(d)$. Then AX_t starting from Ax is identically distributed as X_t starting from $x \in \mathbf{R}^d$ as a stochastic process provided that $Ax = x$, where X is the unique weak solution to (3.1).

Proof. Put $Y = AX$. Then,

$$dY_t = A_0dX_t = A_0\sigma(X_t)dW_t + A_0\mu(X_t)dt,$$

which equals to

$$\sigma(AX_t)U_x^{-1}dW_t + \mu(AX_t)dt,$$

by the assumptions (3.4), where we note that $U_x^{-1}W_t$ is another Wiener process. Namely, Y is a weak solution to

$$dY_t = \sigma(Y_t)dW_t + \mu(Y_t)dt.$$

By the uniqueness of the solution, we have the assertion. \square

As in the one dimensional case, we *symmetrize* both σ and μ to get another diffusion with the PCS for which an extended static hedging relation still holds.

Theorem 3.4. *Suppose that we are given a family of hyperplanes indexed by Φ which satisfies the assumptions of Lemma 2.2. Put*

$$(3.5) \quad \tilde{\sigma}(x) = \sum_{g \in G} T_g \sigma(g^{-1}x) 1_{\{x \in g\Sigma_\Phi\}} U_x, \quad \tilde{\mu}(x) = \sum_{g \in G} T_g \mu(g^{-1}x) 1_{\{x \in g\Sigma_\Phi\}},$$

where T_g is an orthogonal matrix corresponding to the reflection part of g , and $x \mapsto U_x \in O(d)$ is a piecewise continuous map. Let \tilde{X} be a solution to

$$d\tilde{X}_t = \tilde{\sigma}(\tilde{X}_t) dW_t + \tilde{\mu}(\tilde{X}_t) dt.$$

Then for a bounded Borel function f with its support in Σ_Φ , we have

$$(3.6) \quad E[f(X_t) 1_{\{\tau_{\Sigma_\Phi} > t\}}] = \sum_{g \in G} \eta(g) E[f(g^{-1}(\tilde{X}_t))].$$

Here we used the notations set in section 2.3.

Proof. It suffices to show that $\tilde{\sigma}$ and $\tilde{\mu}$ satisfy (3.4). For $h \in G$, we can find $h_0 \in \mathbf{R}^d$ and $T_h \in O(d)$ such that $hx = T_h x + h_0$. We then have

$$\begin{aligned} \tilde{\sigma}(hx) &= \sum_{g \in G} T_h T_h^{-1} T_g \sigma(g^{-1}hx) 1_{\{x \in h^{-1}g\Sigma_\Phi\}} U_{hx} \\ &= T_h \sum_{g \in G} T_{h^{-1}g} \sigma((h^{-1}g)^{-1}x) 1_{\{x \in h^{-1}g\Sigma_\Phi\}} U_{hx} \\ &= T_h \tilde{\sigma}(x) U_{hx}. \end{aligned}$$

Here we have used $T_f T_g = T_{fg}$ for $f, g \in G$, which can be verified by

$$fg(x) = f(T_g x + g_0) = T_f T_g x + T_f g_0 + f_0,$$

where $fx = T_f x + f_0$ and $gx = T_g x + g_0$.

Similarly we have $\tilde{\mu}(hx) = T_h \mu(x)$. \square

Remark 3.5. We can use the numerical scheme based on the equation (3.6) for general Φ since, by a linear transformation, a generic Φ (and given X) can be transformed into the system satisfying the assumptions in Lemma 2.2.

4. APPLICATIONS

4.1. Stochastic volatility models. As we have stated in the previous section, we give a certificate to the use of the PCS symmetrization method to such stochastic volatility models as Heston's and SABR type.

A generic stochastic volatility model is given as follows:

$$(4.1) \quad \begin{aligned} dX_t &= \sigma_{11}(X_t, V_t)dW_t + \mu_1(X_t, V_t) dt \\ dV_t &= \sigma_{21}(V_t)dW_t + \sigma_{22}(V_t)dB_t + \mu_2(V_t) dt, \end{aligned}$$

where W and B are mutually independent (1-dim) Wiener processes, $\mu(x, v) = (\mu_1(x, v), \mu_2(v))$ is a continuous function on \mathbf{R}^2 , and

$$\sigma(x, v) = \begin{pmatrix} \sigma_{11}(x, v) & 0 \\ \sigma_{21}(v) & \sigma_{22}(v) \end{pmatrix}$$

is a continuous function on \mathbf{R}^2 . In most cases, $\sigma_{11}(x, v) = x\nu(v)$ for some ν and $\mu_1(x, v) = rx$ from financial reasoning, and x - and v - axes are set to be natural boundary. In our scheme, however, these features are irrelevant but that V do not depend on S is important.

Corollary 4.1. *Let $K > 0$ and put*

$$\begin{aligned} \tilde{\sigma}_{11}(x, v) &= \begin{cases} \sigma_{11}(x, v) & x \geq K \\ -\sigma_{11}(2K - x, v) & x < K \end{cases} \\ \tilde{\mu}_1(x, v) &= \begin{cases} \mu_1(x, v) & x \geq K \\ -\mu_1(2K - x, v) & x < K \end{cases}, \end{aligned}$$

and let \tilde{X} be the unique (weak) solution to

$$d\tilde{X}_t = \tilde{\sigma}_{11}(\tilde{X}_t, V_t)dW_t + \tilde{\mu}_1(\tilde{X}_t, V_t) dt,$$

where V is the solution to (4.1). Then, it holds for any bounded Borel function f and $t > 0$ that

$$\begin{aligned} &E[f(X_t - K)1_{\{X_t > K\}}1_{\{\tau_K > t\}}] \\ &= E[f(\tilde{X}_t - K)1_{\{\tilde{X}_t > K\}}] - E[f(K - \tilde{X}_t)1_{\{\tilde{X}_t < K\}}], \end{aligned}$$

where X is the solution to (4.1) with $X_0 > K$ and τ_K is the first hitting time of X to K .

Proof. Apply Theorem 3.4 to $\Phi = \{((1, 0), K)\}$, with $U_x \equiv I$. \square

We can also obtain a static hedging formula for a double barrier option. Let K' be a positive constant. We consider static hedging of an option knocked out if X of (4.1) hit either the boundary $x = K$ or $x = K + K'$.

Corollary 4.2. *Set*

$$\begin{aligned} & \hat{\sigma}_{11}(x, v) \\ &= \begin{cases} \sigma_{11}(x - 2nK', v) & K + 2nK' \leq x < k + (2n + 1)K', \\ -\sigma_{11}(2K - x + 2nK', v) & K + (2n - 1)K' \leq x < K + 2nK', \end{cases} \\ & \quad n \in \mathbf{Z}, \end{aligned}$$

$$\begin{aligned} & \hat{\mu}_1(x, v) \\ &= \begin{cases} \mu_1(x - 2nK', v) & K + 2nK' \leq x < k + (2n + 1)K', \\ -\mu_1(2K - x + 2nK', v) & K + (2n - 1)K' \leq x < K + 2nK', \end{cases} \\ & \quad n \in \mathbf{Z}, \end{aligned}$$

and let \hat{X} be the unique (weak) solution to

$$d\hat{X}_t = \hat{\sigma}_{11}(\hat{X}_t, V_t)dW_t + \hat{\mu}_1(\hat{X}_t, V_t)dt,$$

where V is the solution to (4.1). Then, it holds for any bounded Borel function f and $t > 0$ that

$$\begin{aligned} & E[f(X_t - K)1_{\{X_t \in (K, K+K')\}}1_{\{\tau_{(K, K+K')} > t\}}] \\ &= \sum_{n \in \mathbf{Z}} E[f(\hat{X}_t - K - 2nK')1_{\{\hat{X}_t \in (K, K+K')\}}] \\ & \quad - \sum_{n \in \mathbf{Z}} E[f(K - \hat{X}_t + (2n - 1)K')1_{\{\hat{X}_t \in (K, K+K')\}}], \end{aligned}$$

where X is the solution to (4.1) with $X_0 \in (K, K + K')$ and $\tau_{(K, K+K')}$ is the first exit time of X out of $(K, K + K')$.

Proof. Apply Theorem 3.4 to $\Phi = \{((1, 0), K), ((-1, 0), K + K')\}$, with $U_x \equiv I$. \square

4.2. Time-inhomogeneous extension. The trick of Corollary 4.1 further leads to a time-inhomogeneous extension of Theorem 3.4. Even more generally, we have the following

Theorem 4.3. *We reset X and V in (4.1) as follows: X is \mathbf{R}^{d_1} valued and V is \mathbf{R}^{d_2} -valued, with $\sigma_{11} : \mathbf{R}^{d_1} \times \mathbf{R}^{d_2} \rightarrow \mathbf{R}^{d_1} \otimes \mathbf{R}^{d_1}$, $\sigma_{2i} : \mathbf{R}^{d_2} \rightarrow \mathbf{R}^{d_2} \otimes \mathbf{R}^{d_i}$, $i = 1, 2$, $\mu_1 : \mathbf{R}^{d_1} \otimes \mathbf{R}^{d_2} \rightarrow \mathbf{R}^{d_2}$, and $\mu_2 : \mathbf{R}^{d_2} \rightarrow \mathbf{R}^{d_2}$, all piecewise continuous, and W , d_1 -dimensional, and B , d_2 -dimensional, are mutually independent Wiener processes. For the SDE for V , we simply assume that $\sigma_{2i}, i = 1, 2$ and μ_2 are given so that there is a unique strong solution. Suppose that $\Phi_0 \subset \mathbf{R}^{d_1} \otimes \mathbf{R}$ satisfies the assumptions*

of Lemma 2.2. Let G be the reflection group associated with Φ . Then we still have (3.6)¹.

Proof. Set $\Phi = \{((\alpha, 0), k) \in \mathbf{R}^{d_1+d_2} : (\alpha, k) \in \Phi_0\}$, and apply Theorem 3.4. \square

Corollary 4.4. *Theorem 3.4 is generalized to time-inhomogeneous cases: σ and μ can be time-dependent.*

Proof. Take $d_2 = 1$ and $\sigma_{21} = \sigma_{22} = 0$ (and $\mu_2 = 1$) in Theorem 4.3. \square

4.3. Curved boundary. Let $D \subset \mathbf{R}^d$ be a domain homeomorphic to a Σ_Φ satisfying the assumptions in Lemma 2.2. The scheme can be applied to the numerical calculation of such an expectation as

$$E[f(X_t)1_{\{X_t \in D\}}1_{\{\tau_D > t\}}],$$

where

$$\tau_D := \inf\{t > 0 : X_t \notin D\}.$$

In fact, at least heuristically, $F : D \rightarrow \Sigma_\Phi$ being a homeomorphism, $Y_t = F(X_t)$ can be realized as a solution to a time-inhomogeneous SDE in Σ_Φ , and we can apply Corollary 4.4. Note that in the scheme the information of the outside of Σ_Φ of the coefficients of SDE is totally irrelevant.

4.4. Time-dependent boundary. The scheme is also applicable to the cases where knock-out boundaries are time-dependent.

Let $\alpha_i : [0, \infty) \rightarrow \mathbf{R}^d$, $i = 1, \dots, d$ be smooth maps such that the matrix $A(t) = [\alpha_1(t), \dots, \alpha_d(t)]$ is invertible for all t and is at most linear growth in t . For each $t \geq 0$, put $\Phi(t) = \{(\alpha_i(t), k_i)\}^2$ and let $\Sigma(t)$ be the intersection of the half spaces associated with $\Phi(t)$. We will briefly discuss how it can be transformed to the previous problem, though heuristically.

Let $C : [0, \infty) \rightarrow \mathbf{R}^d \otimes \mathbf{R}^d$ be a solution (unique up to the initial point) to the following matrix valued linear differential equation:

$$(4.2) \quad C'(t) = C_t A'(t) \{A(t)\}^{-1}.$$

Then $C(t)\alpha_i(t) = C(0)\alpha_i(0) =: \alpha_i$ for $i = 1, 2, \dots, d$ since (4.2) is equivalent to

$$(C(t)A(t))' = 0.$$

Moreover, since

$$\langle \alpha_i(t), x \rangle = \langle C(t)\alpha_i(t), C^*(t)^{-1}x \rangle = \langle \alpha_i, C^*(t)^{-1}x \rangle,$$

¹ Precisely speaking, the group G should be embedded in Affine group in $\mathbf{R}^{d_1+d_2}$

² Without loss of generality, we may assume that k_i , $i = 1, 2, \dots, d$ are constants.

we see that $x \mapsto C^*(t)^{-1}x$ transforms $\Sigma(t)$ to a time-independent domain Σ_{Φ^*} with $\Phi^* = \{(\alpha_i, k_i), i = 1, 2, \dots, d\}$. Now the problem turns into the pricing of options written on $C^*(t)^{-1}X_t$ knocked out at the boundary of Σ_{Φ^*} . As we have commented in Remark 3.5, we can assume that Φ^* satisfies the assumptions in Lemma 2.2. Combining them all, we see that Corollary 4.4 is now applicable.

REFERENCES

- [1] Carr, P. and Lee, R, “Put-Call Symmetry: Extensions and Applications”, *Mathematical Finance*, Vol.19, No.4 (October 2009), 523-560.
- [2] Humphreys, J. E. (1990). *Reflection groups and Coxeter groups*. Cambridge Studies in Advanced Mathematics, 29, Cambridge University Press, Cambridge.
- [3] Imamura, Y., Ishigaki, Y., Kawagoe, T. and Okumura, T. “A Numerical Scheme Based on Semi-Static Hedging Strategy”, 2012, preprint.
- [4] Imamura, Y., Ishigaki, Y., Kawagoe, T. and Okumura, T. “Some Simulation Results of the Put-Call Symmetry Method Applied to Stochastic Volatility Models”, Proceedings of the 43rd ISCIE International Symposium on Stochastic Systems Theory and Its Application, 2012.
- [5] Imamura, Y., Ishigaki, Y., and Okumura, T. “The Put-Call Symmetry Method Applied to Stochastic Volatility Models”, 2012, preprint.
- [6] Imamura, Y. and Takagi, K. “Semi-Static Hedging Based on a Generalized Reflection Principle on a Multi Dimensional Brownian Motion”, 2012, preprint, to appear in *Asia-Pacific Financial Markets*.

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